Pingala - I: The Problem

Here are a few consecutive numbers and their 'prime decomposition' (that is, the number *molecule* expressed as prime *atoms* when multiplied...) . As you can guess, this decomposition cannot be reduced any further. This is known as the Fundamental Theorem of Arithmetic.

$55 = 5.11$
$56 = 2^3.7$
$57 = 3.19$
$58 = 2.29$
$59 = 59$ (prime)
$60 = 2^2.3.5$
$61 = 61$ (prime)

As you will notice, two consecutive numbers have no factors in common. This is useful information for us. This suggests that there could be a relationship between the prime factors of two consecutive numbers. If we know the prime formula of N, we could say what it would be for (N+1) without any trial divisions. Alas, but such an algorithm or theorem has not been discovered yet. There is something fundamental we do not understand here, *the relationship between addition and multiplication*.

A property of numbers that contains both addition and multiplication is called the sum of divisors, or σ(*sigma*). For a prime number there are only two divisors, which we write in symbols thus:

$\sigma(59) = 1 + 59 = 60$.

For composite numbers, the calculation gets more and more complicated - depending on the number of primes in their decomposition. To show how both addition and multiplication of primes play a role, let us see a couple of examples:

$$\sigma(55) = 1 + 5 + 11 + 55 = 72$$
$$\sigma(56) = 1 + 2 + 2^2 + 2^3 + 7 + 7.2 + 7.2^2 + 7.2^3 = 120$$

A divisor is one particular multiplicative combination of prime factors, out of all the possible ones...and when all these combinations are added, we get the sum of divisors for n.

Can we deduce the value $\sigma(56)$=120, from the decomposition of the previous number i.e, $55 = 5.11$? The question is simple but it shows, that from the perspective of prime factors - the addition of 1 to 55 (making it 56) is a very deep question. In fact, there are many ways to look at what information $\sigma(n)$ can provide about $\sigma(n+1)$. We will isolate a special situation where $\sigma(n) = \sigma(n+1)$. One can check by hand calculation that:

$$\sigma(14) = \sigma(15) = 24$$

This is the first number for which our condition holds true. Next we have $\sigma(206) = \sigma(207) = 312$ and that is already very time consuming, unless you do it on a computer. The next few numbers in this sequence are:

$$14, 206, 957, 1334, 1364, 1634, 2685, 2974, 4364, 14841, 18873, 19358, 20145, 24957, 33998, 36566, 42818, 56564, 64665, 74918, 79826, 79833, 84134, 92685, 109214, 111506, 116937, 122073, 138237, 147454, 161001, 162602, 166934....$$

We will now state the problem. In simple words, we want you to find a pattern in the sequence above. A rigorous proof will be appreciated but is not required. We just want to see a logical pattern. The task then is to find:

a function or algorithm f(n) such that for values of n=1,2,3,4.... its output is the sequence above.
a function or algorithm f(n) where n is one of the numbers in the sequence and the output of f(n) is the next number in the sequence.

The terms above are merely indicative, what we really want is a breakthrough insight into this fundamental problem. As an example of our approach, we did an experiment. Given a number k...we tried to find by computer for how many n, $\sigma(n) = k$ under an upper limit to n (see reference 5). We could not find anything conclusive, however.

Whoever helps us understand and solve this problem will have earned the prize, which is the first in a series of contests.

Historical Note

This sequence is labelled A002961 in the Online Encyclopaedia of Integer Sequences. It is not known whether this sequence goes on forever, or terminates. In that form, the question has been called the ***Erdös-Sierpiński problem***. Professional mathematicians will remember this as Problem B13 in the book *Unsolved Problems In Number Theory* by Richard Guy. The divisor function $\sigma(n)$ was also studied by Ramanujan, who proved a result in 1915 related to Robin's Theorem, under the assumption that the Riemann Hypothesis is true. We hope that any progress with this problem will reveal important information about one of the major goals of ZetaTrek.

References

1. The Prime Facts: From Euclid to AKS, by Scott Aaronson. This paper is short and very suitable for beginners to the subject.
2. Primes Is In P: A Breakthrough For Everyman, by Folkmar Bornemann
3. A002961, Online Encyclopaedia of Integer Sequences
4. Solutions of the problem of Erdos-Sierpinski: by Lourdes Benito
5. The Sum of All Divisors: Experiments with GAP + Gnuplot